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Generalization formula for the notable limits of Euler's number (also known as Napier's constant)

Andrea Nocco

Abstract

The Euler/Napier's number limits are important expressions used in mathematical analysis. They are useful, for example, in solving indeterminate forms like 1^∞. The most common equation is the general definition of the Euler's number:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \tag{1}$$

This expression is often found in many other variations, like: $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$, $\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x = \frac{1}{e}$, $\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^{mx} = e^{mk}$, $\lim_{x\to\infty}\left(\frac{x}{x+k}\right)^x = \frac{1}{e^k}$ ....

The goal of this article is to find a more generalized expression of (1), in order to embrace all variations. The final result is described as:

Let $\overline{\mathbb{R}} = \mathbb{R}\cup\{\pm\infty\}$ and $x_0 \in \overline{\mathbb{R}}$, if $f(x)$ is *infinitesimal* for $x\to x_0$, then:

$$\lim_{x\to x_0}(1+af(x))^{bg(x)} = e^{abl} = e^{ab\lim\limits_{x\to x_0}(f(x)g(x))} \tag{2}$$

If $a, b = 1$, it reduces into:

$$\lim_{x\to x_0}(1+f(x))^{g(x)} = e^l \tag{3}$$

where $l = \lim\limits_{x\to x_0} f(x)g(x)$.

Contents

Chapter 1

Generalization formula for the Euler/Napier's limit

The *infinte* and *infinitesimal* functions definitions are required in order to understand the generalization formula.

1.1 *Infinite* and *infinitesimal* functions

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and $x_0 \in \overline{\mathbb{R}}$ accumulation point for A, with $f(x) \neq 0$ around x_0, then:

- f is ***infinitesimal*** for $x \to x_0$ if $\exists \lim\limits_{x \to x_0} |f(x)| = 0$

- f is ***infinite*** for $x \to x_0$ $x \to x_0$ if $\exists \lim\limits_{x \to x_0} |f(x)| = +\infty$

The same notion can be extended to *infinitesimal* and *infinite* functions for $x \to x_0^-$ or $x \to x_0^+$ (De Mitri 2013).

1.1.1 Comparing *infinite* and *infinitesimal* functions

Let f and g *infinitesimal* for $x \to x_0$, then:

- f is *infinitesimal* of a greater order than g if $\exists \lim\limits_{x \to x_0} \left|\frac{f(x)}{g(x)}\right| = 0$.

Let f and g *infinite* for $x \to x_0$, then:

- f is *infinite* of a greater order than g if $\exists \lim\limits_{x \to x_0} \left|\frac{f(x)}{g(x)}\right| = +\infty$.

When f and g are both *infinitesimal* or both *infinite* for $x \to x_0$, then:

- f and g are *infinitesimal* or *infinite* of the same order if $\exists \lim_{x\to x_0} \left|\frac{f(x)}{g(x)}\right| \in \mathbb{R}^+$, or more in general if $\exists h, k \in \mathbb{R}^+$ such that $h \leq \left|\frac{f(x)}{g(x)}\right| \leq k$ around x_0.

Last, when f and g are both *infinitesimal* or both *infinite* for $x \to x_0$, but none of the previous conditions apply:

- f and g are *infinitesimal* or *infinite* of not comparable order.

1.2 Asymptotic equivalence

f is ***equivalent*** or ***asymptotic*** to g for $x \to x_0$ (in symbols $f(x) \simeq g(x)$ for $x \to x_0$), if $\exists \lim_{x\to x_0} \left|\frac{f(x)}{g(x)}\right| = 1$, or in other words if $f(x) = g(x) + o(g(x))$ for $x \to x_0$.
From this definition, derives that x and $ln(1+x)$ are asymptotically equivalent for $x \to x_0$, since $\lim_{x\to x_0} \frac{x}{ln(x+1)} = 1$. Furthermore x and $ln(1+x)$ are *infinitesimal* of the same order for $x \to x_0$.
Generalizing the last formula, for every *infinitesimal* function $f(x)$, for $x \to x_0$, then:

$$ln(x+1) \simeq f(x), \tag{1.1}$$

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1.3 General Napier's limit definition

Let $f, g : A \subseteq \mathbb{R} \to \mathbb{R}$ with f *infinitesimal* for $x \to x_0 \in \overline{\mathbb{R}}$ and $a, b \in \mathbb{R}$, then:

$$\boxed{\lim_{x\to x_0} (1 + af(x))^{bg(x)} = e^{abl} = e^{ab \lim_{x\to x_0} f(x)g(x)}} \tag{1.2}$$

If $a, b = 1$, then 1.3.3 becomes:

$$\boxed{\lim_{x\to x_0} (1 + f(x))^{g(x)} = e^l}$$

where $l = \lim_{x\to x_0} f(x)g(x)$.

1.3.1 Demonstration

Using logarithmic properties, left term of 1.3.3 becomes:

$$\lim_{x\to x_0} (1 + af(x))^{bg(x)} = \lim_{x\to x_0} e^{ln\left[(1+af(x))^{bg(x)}\right]} = e^{\lim_{x\to x_0}[bg(x)ln(1+af(x))]} \tag{1.3}$$

Because of 1.1, then:

$$e^{\lim\limits_{x\to x_0}[bg(x)ln(1+af(x))]} = e^{ab\lim\limits_{x\to x_0}[g(x)f(x)]} = e^{abl} \tag{1.4}$$

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1.3.2 Corollary: outcome of 1.3.3

Let $f, g : A \subseteq \mathbb{R} \to \mathbb{R}$, if f and g are *infinitesimal* or *infinite* of the same order, then:

$$e^{abl} \in \mathbb{R}^+ \implies \lim_{x\to x_0}(1+af(x))^{bg(x)} \in \mathbb{R}^+ \tag{1.5}$$

So the result of this Napier's limit is always a positive number.

1.3.3 Other examples

Definition 1.3.3 is useful to calculate for example the following limits:

$$\lim_{x\to 0}(1+(1-\cos x))^{\frac{1}{x^2}} = e^{\frac{1}{2}} = \sqrt{e}$$

since $l = \lim\limits_{x\to 0}\frac{1-\cos x}{x^2} = \frac{1}{2}$

Other examples can be:

$$\lim_{x\to 0}(1+3x)^{\frac{1}{\sin x}} = e^3$$

$$\lim_{x\to +\infty}\left(\frac{x+a}{x+b}\right)^{cx} = e^{(a-b)c}$$

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References

De Mitri, Cosimo (2013). *Analisi matematica III*. Università del Salento.